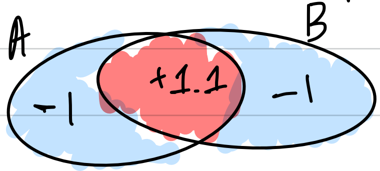
Math 564: Advance Analysis 1
Lecture 18

Jordan decomp. theorem follows from Hahn decomposition.

Def. Let ν be a signed measure on a measurable space $(X, \mathcal{B})$.
Call a set $A \subseteq X$ *ν-positive* if $A \in \mathcal{B}$ and $\nu(A) \geq 0$.
Call a set $A \subseteq X$ *ν-purely positive* if A is ν-positive and every subset of A in $\mathcal{B}$ is ν-positive. We define *ν-negative* and *ν-purely negative* analogously, and we drop ν from the terminology if it is clear.

Caution. The union of two non-disjoint positive sets may not be positive:



$\nu(A \cup B) = 1.1 - 2 = -0.9$,
while $\nu(A) = \nu(B) = 0.1$.

However:

Obs. (a) Positive/negative sets are closed under ctbl disjoint unions.
(b) Purely positive/negative sets are closed under ctbl (not necessarily disjoint) unions.

Hahn decomposition theorem. Let ν be a signed measure on a measurable space $(X, \mathcal{B})$. Then $\exists$ unique (up to purely pos. null sets) partition $X = X_+ \sqcup X_-$, where X_+ is purely positive and X_- is purely negative. In particular, $\nu = \nu|_{X_+} - \nu|_{X_-}$ is a Jordan decomposition.
Uniqueness: if $X = \tilde{X}_+ \sqcup \tilde{X}_-$ is another such partition, then $X_+ \Delta \tilde{X}_+$ and $X_- \Delta \tilde{X}_-$ are purely positive null.

Proof. WLOG, suppose $\nu < \infty$ (otherwise replace it with $-\nu$).

Claim. Every non-null positive set $P \subseteq X$ contains a purely positive subset $P_+ \subseteq P$ with $\nu(P_+) \geq \nu(P) > 0$.

Proof. We prove this by a $\frac{1}{2}$-measure exhaustion argument.
Let $N_0 \subseteq P$ be $\frac{1}{2}$-largest negative set, i.e.

$$-\nu(N_0) \geq \frac{1}{2} \overline{\sup} \{-\nu(N) : N \subseteq P \text{ and } N \text{ is negative}\},$$

where $\overline{\sup} := \min\{1, \sup\}$. Then, $P \setminus N_0$ is still positive, so we repeat: given a disjoint sequence $(N_i)_{i<k}$, let N_k be $\frac{1}{2}$-largest negative subset of $P \setminus \bigcup_{i<k} N_i$, i.e.

$$-\nu(N_k) \geq \frac{1}{2} \overline{\sup} \{-\nu(N) : N \subseteq P \setminus \bigcup_{i<k} N_i, N \text{ negative}\}.$$

Having obtained a pairwise disjoint sequence $(N_k)_{k \in \mathbb{N}}$, let $P_+ = P \setminus \bigcup_{k \in \mathbb{N}} N_k$. Note that by ctbl additivity, $\nu(P) = \nu(P^+) + \sum_{k \in \mathbb{N}} \nu(N_k)$, so $\nu(P^+) \geq \nu(P)$ and $\sum_{k \in \mathbb{N}} -\nu(N_k) < \infty$. It follows that P^+ is purely positive because if $\exists$ non-null negative $N \subseteq P^+$ then because $(-\nu(N_k))$ is summable, $\exists k$ s.t. $-\nu(N_k) < \frac{1}{2} \min\{1, -\nu(N)\}$, contradicting the choice of N_k. $\boxed{\text{Claim}}$

We know "collect all" purely positive sets, so that what's left is purely negative. We do so by $\frac{1}{2}$-measure exhaustion argument. Let P_0 be a $\frac{1}{2}$-largest purely positive set, i.e.

$$\nu(P_0) \geq \frac{1}{2} \overline{\sup} \{\nu(P) : P \subseteq X \text{ purely positive}\}.$$

... Suppose $(P_i)_{i<k}$ is a disjoint sequence of purely positive sets, and let P_k be $\frac{1}{2}$-largest purely positive subset of $X \setminus \bigcup_{i<k} P_i$, i.e.

$$\nu(P_k) \geq \frac{1}{2} \overline{\sup} \{\nu(P) : P \subseteq X \setminus \bigcup_{i<k} P_i \text{ purely positive}\}.$$

Let $X_+ := \bigcup_{k \in \mathbb{N}} P_k$ and $X_- := X \setminus X_+$. Then X_+ is purely positive and $\nu(X_+) < \infty$ by our assumption, so $\sum_{k \in \mathbb{N}} \nu(P_k) = \nu(X_+) < \infty$.

It now follows that X_- is purely negative because otherwise, $\exists$ non-null positive $P \subseteq X_-$, so $\exists$ purely positive $P_+ \subseteq P$ with $\nu(P_+) \geq \nu(P) > 0$, hence $\exists k$ s.t. $\nu(P_k) < \frac{1}{2} \min\{1, \nu(P_+)\}$, contradicting the choice of P_k. $\square$

Cor. For a signed measure ν, if $\nu < \infty$, then $\exists M > 0$ s.t. $\nu \leq M$.

Proof. Let $M := \nu(X_+)$. $\square$

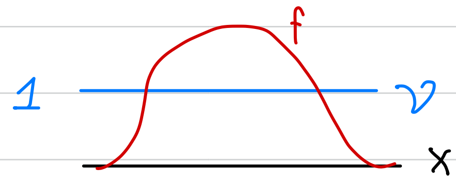
Recall we wanted to prove:

Lebesgue decomposition theorem. For any two σ-finite measures μ, ν on a measurable space $(X, \mathcal{B})$, $X := X_1 \sqcup X_0$ s.t. $\mu|_{X_1} \ll \nu|_{X_1}$ and $\mu|_{X_0} \perp \nu|_{X_0}$.

One can prove this directly without signed measures by a $\frac{1}{2}$-measure exhaustion argument, but we will just prove a stronger theorem:

Lebesgue-Radon-Nikodym theorem. For any two σ-finite measures μ, ν on a measurable space $(X, \mathcal{B})$, there is a partition of $X = X_1 \sqcup X_0$ into subsets in $\mathcal{B}$ s.t. $\mu|_{X_0} \perp \nu|_{X_0}$ and $\mu|_{X_1} \ll \nu|_{X_1}$, moreover, $d(\mu|_{X_1}) = f\, d(\nu|_{X_1})$ for some non-negative $\mathcal{B}$-measurable $f : X \to [0, \infty)$, i.e. $\forall B \subseteq X_1$, $B \in \mathcal{B}$, we have

$$\mu(B) = \int_B f \, d\nu.$$



X_0, X_1 and f are unique up to null sets. This f is called the **Radon-Nikodym derivative** of $\mu|_{X_1}$ over $\nu|_{X_1}$ and is denoted $\frac{d(\mu|_{X_1})}{d(\nu|_{X_1})}$.

To prove this, we need the following lemma:

Lemma. For finite measures μ, ν on a measurable space $(X, \mathcal{B})$,
either: $\mu \perp \nu$
or: $\mu|_A \geq \varepsilon \cdot \nu|_A$ for some $A \in \mathcal{B}$ with $\nu(A) > 0$,
and for some $\varepsilon > 0$.

Proof. Let $\varepsilon_n := \frac{1}{n}$. Let $P_n \sqcup N_n$ be the Hahn decomposition of X for the signed measure $\mu - \varepsilon_n \cdot \nu$. Let $P := \bigcup\limits_{n \in \mathbb{N}} P_n$ and $N = X \setminus P = \bigcap\limits_{n \in \mathbb{N}} N_n$. Note that $0 \leq \mu(N) \leq \varepsilon_n \nu(N)$ for all n, so

$\mu(N) = 0$. If $\nu(P) = 0$ then $X = P \sqcup N$ witnesses $\mu \perp \nu$.
Otherwise, $\exists$ some n s.t. $\nu(P_n) > 0$ and $\mu|_{P_n} - \varepsilon_n \cdot \nu|_{P_n} \geq 0$ $\square$

Proof of Lebesgue-Radon-Nikodym. The uniqueness is clear (if there is another $\tilde{f}$, then $\int |f - \tilde{f}| \, d\nu = 0$, so $f = \tilde{f}$ ν-a.e.).
First assume μ, ν are finite.